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TOWARD A GENERAL THEORY OF PLATES AND SHELLS

WITH FINITE DISPLACEMENTS AND STRAINS

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In the first two sections of the present paper a short presentation of the nonlinear theory of shells is given using the usual asymmetric tensors of the tangential forces and moments $T_*^{\alpha\beta}$ and $H^{\alpha\beta}$. For small strains in a thin shell these tensors can be considered to be symmetrical to the accuracy of the Kirchhof hypothesis; i.e., $T_*^{\alpha\beta} = T_*^{\beta\alpha}$, $H^{\alpha\beta} = H^{\beta\alpha}$. For the case of a shell of medium thickness, this simplification leads to an error of the order $\sqrt{h/R}$ compared to unity, and to an even higher error for shells made of material not obeying a linear Hooke's law. If the force and moment tensors are not symmetrical, excessive difficulties arise in formulating the nonlinear relations of elasticity, the general theorems of the nonlinear theory of shells, the evaluation of the accuracy of apprimation methods, etc. Therefore in Section 3 of this paper, the basic force and moment tensors are introduced in a form symmetric for arbitrary strains. In this case it is quite natural to assume a distribution of tangential stresses over the thickness of the shell rather than to make geometrical assumptions. Using the symmetrical force and moment tensors it is possible to find a general expression for the strain potential and to express the general integrals by homogeneous equilibrium equations, not through four functions as is usually done, but by three.

In Section 4 the general equations of elasticity for isotropic shells are derived.

In Section 5 it is proved that the Galerkin equations in the theory of finite strains is not directly connected with the principle of minimum potential energy as is the case in the linear theory.

In Section 6 a functional R is introduced which has a stationary value when static values of the boundary conditions and the equations of equilibrium are reached. For the simplified equations the corresponding functional is considered in Reference 1. Since the Kirchhof hypothesis is not used to obtain equation (6.22) and the strains are considered to be arbitrary, the variational equation $\delta R = 0$ is applicable to a shell of medium thickness as well as to physically nonlinear problems.

In the functional of (6.22), in addition to the forces and moments, the quantities $e_{\alpha\beta}$ and ω_α; these satisfy three geometrical identities, which do not appear as simultaneous conditions for finite strains. Therefore an adjacent stress condition is not required to satisfy the conditions of continuity of finite strains and the variational principle $\delta R = 0$ is analogous in a physical sense to the Castigliano principle in the linear theory of elasticity.

In Section 7 the functional R is transformed to the form (7.11), which does not contain a displacement.

1. <u>First and Second Strain Tensors of a Surface</u>. We refer the median surface S of an unstrained shell to Gaussian coordinates x^1 and x^2 and introduce the following notations: ρ is the radius vector of the point (x^1, x^2), ρ_α are the coordinate vectors of the surface S, m is the unit vector normal to S at the point (x^1, x^2), $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the components of the first and second metric tensors of the surface S, so that

$$\rho_\alpha = \frac{\partial \rho}{\partial x^\alpha},\ a_{\alpha\beta} = \rho_\alpha \cdot \rho_\beta,\ b_{\alpha\beta} = -m_\alpha \cdot \rho_\beta = -\frac{\partial m}{\partial x^\alpha} \cdot \rho_\beta$$

$c_{\alpha\beta}$ are the components of the discriminant tensor:

$$c_{\alpha\beta} = (\rho_\alpha \times \rho_\beta)m,\ c_{\alpha\alpha} = 0,\ c_{12} = -c_{21} = \sqrt{a},\ a = a_{11}a_{22} - a_{12}^2$$

For the right-hand oriented trihedron $\rho_1 \rho_2 m$ we have

$$\rho_\alpha \times \rho_\beta = c_{\alpha\beta}\, m,\ m \times \rho_\alpha = c_{\alpha\beta}\, \rho^\beta \tag{1.1}$$

where $\rho_\beta = a^{\alpha\beta} \rho_\alpha$ are the vectors of the reciprocal basis $a^{\alpha\beta} = c^{\alpha\pi} c^{\beta\lambda} a_{\pi\lambda}$.

For the vectors ρ_α and m the following equations are valid:

$$\nabla_\alpha \rho_\beta = b_{\alpha\beta}\, m,\ \nabla_\alpha \rho^\beta = m b^\beta_\alpha,\ m_\alpha = -b^\beta_\alpha \rho_\beta \tag{1.2}$$

Here and in the following, ∇ is a sign for covariant differentiation with respect to $a_{\alpha\beta}$.

Let C be the contour of the median surface, s its arc length, n and τ unit vectors normal and tangent to C. The components of n and τ in the system of coordinates x^1 and x^2 on S are given by the formulas

$$n_\alpha ds = c_{\alpha\beta}\, dx^\beta, \quad \tau^\alpha ds = dx^\alpha, \tag{1.3}$$
$$n^\alpha = c^{\alpha\beta}\tau_\beta, \quad \tau_\beta = c^{\alpha\beta} n_\alpha$$

In the following, all geometrical and physical quantities referring to the strained surface S* will be starred. The formulas given above hold also for the strained surface if ρ, ρ_α, m, $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$, n_α, τ_α, ds, ... are changed to $\rho*$, $\rho_\alpha *$, m_*, $a_{\alpha\beta}*$, $b_{\alpha\beta}*$, $c_{\alpha\beta}*$, $n_\alpha *$, $\tau_\alpha *$, ds*, ...

Let v be the displacement vector of the point (x^1, x^2) of the surface S, $\rho * = \rho + v$ is the radius vector of this point after strain deformation, i.e., of the point (x^1, x^2) of the surface S*. The components of the first tensor (p) and the second tensor (q) of strain are determined by the equations[2]

$$2p_{\alpha\beta} = a_{\alpha\beta}* - a_{\alpha\beta} = e_{\alpha\beta} + e_{\beta\alpha} + a^{\pi\lambda} e_{\alpha\pi} e_{\beta\lambda} + \omega_\alpha \omega_\beta, \tag{1.4}$$
$$q_{\alpha\beta} = b_{\alpha\beta}* - b_{\alpha\beta}$$

$$b_{\alpha\beta}* = -m_\alpha * \cdot \rho_\beta * = \sqrt{\frac{a}{a_*}}\, [E(b_{\alpha\beta} + B_{\alpha\beta}) + E_\gamma \Omega^\gamma_{\alpha\beta}] \tag{1.5}$$

where

$$e_{\alpha\beta} = \nabla_\alpha v_\beta - w b_{\alpha\beta}, \quad \omega_\alpha = \nabla_\alpha \omega + b^\lambda_\alpha v_\lambda, \tag{1.6}$$
$$v = \rho^\alpha v_\alpha + mw$$

$$B_{\alpha\beta} = \nabla_\alpha \omega_\beta + e^\gamma_{\beta\cdot}\, b_{\alpha\beta}, \quad \Omega^\gamma_{\alpha\beta} = \nabla_\alpha e^\gamma_{\beta\cdot} - \omega_\beta b^\gamma_\alpha \tag{1.7}$$

$$2E = c^{\alpha\beta} c_{\pi\lambda} (\delta^\pi_\alpha + e^\pi_{\alpha\cdot})(\delta^\lambda_\beta + e^\lambda_{\beta\cdot}), \tag{1.8}$$
$$E_\alpha = c^{\pi\nu} c_{\alpha\beta}\, \omega_\gamma (\delta^\beta_\pi + e^\beta_{\pi\cdot})$$

$$a* = a(1 + 2p_1 + 4p_2), \quad p_1 = a^{\alpha\beta} p_{\alpha\beta}, \quad 2p_2 = c^{\alpha\beta} c^{\pi\lambda} p_{\alpha\pi} p_{\beta\lambda} \tag{1.9}$$

For the coefficients of the connectivity $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma^{*\gamma}_{\alpha\beta}$ of the surfaces S and S* we have the relations

$$\Gamma^{*\gamma}_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + a_*^{\gamma\lambda} p_{\lambda,\alpha\beta} \quad (p_{\lambda,\alpha\beta} = \nabla_\alpha p_{\beta\lambda} + \nabla_\beta p_{\alpha\lambda} - \nabla_\lambda p_{\alpha\beta}) \tag{1.10}$$

The tensor $p_{\gamma,\alpha\beta}$ depends on the angles of rotation of an element of the median surface. In fact, putting $\nabla_\alpha^* \rho_\beta^* = b_{\alpha\beta}^* m_*$ (where ∇^* is a sign of covariant differentiation with respect to $a_{\alpha\beta}^*$) in the expression

$$\nabla_\alpha \rho_\beta{}^* = m_* b_{\alpha\beta}{}^* + a_*{}^{\gamma\lambda} p_{\lambda,\alpha\beta} \rho_\gamma{}^*$$

we have

$$p_{\gamma,\alpha\beta} = \rho_\gamma{}^* \cdot \nabla_\alpha \rho_\beta{}^* = \rho_\gamma{}^*(\nabla_\alpha \rho_\beta + \nabla_\alpha \nabla_\beta v) =$$
$$(\rho_\gamma + \rho_\gamma e^{\lambda}_{\gamma\cdot} + m\,\omega_\gamma) \cdot (m b_{\alpha\beta} + \nabla_\alpha \nabla_\beta v)$$

Hence, we find

$$p_{\gamma,\alpha\beta} = \omega_\gamma (b_{\alpha\beta} + B_{\alpha\beta}) + \Omega^{\lambda}_{\alpha\beta\cdot}(a_{\gamma\lambda} + e_{\gamma\lambda}) \tag{1.11}$$

2. __The Equations of Equilibrium.__ Let the shell be in a state of equilibrium under the action of the given forces. We subject it to an infinitesimally small possible perturbation characterized by the vector δu and consistent with the constraints present on the shell. Then the initially possible perturbations for a three-dimensional body are expressible by the relations

$$\delta A = \iiint_\Omega F_* \cdot \delta u\, d\Omega + \iint_\Pi p \cdot \delta u\, d\Pi = \iiint_\Omega \sigma^{ik} \delta \varepsilon_{ik}\, d\Omega = \tag{2.1}$$
$$\iiint_\Omega \sigma^{ik} \bar{r}_i{}^* \zeta_i{}^* \cdot \frac{\partial \delta u}{\partial x^k}\, d\Omega$$

Here F^* is the force referred to a unit volume Ω of the strained body, p is the surface force occurring on a unit strained surface Π, and the ε_{ik} are components of the tensor of the finite strains.

Applying (2.1) to the strained shell, we get

$$\delta A = \iint_{\sigma^*} (X_* \cdot \delta v + C_*{}^{\alpha\beta} \rho_\alpha{}^* M_\beta{}^* \cdot \delta m_*)\, d\sigma^* + \tag{2.2}$$

$$\int_{C^*} (\Phi_* \cdot \delta v + C_*{}^{\alpha\beta} \rho_\alpha{}^* \cdot C_* \tau_\beta{}^* \delta m_*)\, ds^* =$$
$$\iint_{\sigma^*} (T_*{}^{\alpha\beta} \delta p_{\alpha\beta} + {}^*b^{\alpha}_{\gamma} H^{\gamma\beta} \delta p_{\alpha\beta} - H^{\alpha\beta} \delta q_{\alpha\beta})\, d\sigma^*$$

Here $T_*{}^{\alpha\beta}$ and $H^{\alpha\beta}$ are the components of the tensor of the tangential forces and moments:

$$T_*^{\alpha\beta} = \int_{-h(-)}^{h(+)} \sqrt{\frac{g*}{a*}}\,\sigma^{\alpha\lambda}(\delta_\lambda^\beta - zb*_\lambda^\beta)dz, \quad N^\alpha = \int_{-h(-)}^{h(+)} \sqrt{\frac{g*}{a*}}\,\sigma^{\alpha 3}dz \qquad (2.3)$$

$$M^{\alpha\beta} = \int_{-h(-)}^{h(+)} \sqrt{\frac{g*}{a*}}\,\sigma^{\alpha\lambda}(\delta_\lambda^\beta - z*b_\lambda^\beta)z\,dz$$

Further, z is a coordinate normal to $S*$; $h(+)$, $h(-)$ are the equations of the boundary surfaces; X_*, M_* are the vectors of the external forces and moments with components X_*^α and M_*^α in the coordinate system of the strained surface; $X_*^3 = X_* m_*$ are referred to a unit area of the strained surface σ^*; Φ^* is the vector of the contour of the load referred to a unit length of the strained contour C_*; G^* is the bending moment on this contour.

From the relations (2.2) we get the equations of equilibrium

$$\nabla_\alpha {}^*T_*^{\alpha\beta} - {}^*b_\alpha^\beta N_*^\alpha + X_*^\beta = 0, \quad \nabla_\alpha {}^*N_*^\alpha + b_{\alpha\beta}\,{}^*T_*^{\alpha\beta} + \qquad (2.4)$$
$$X_*^3 = 0, \quad \nabla_\alpha {}^*M^{\alpha\beta} - N_*^\beta - M_*^\alpha a_*^{\gamma\beta} c_{\alpha\gamma}{}^* = 0$$

or in vector form

$$\nabla_\alpha {}^*f_*^\alpha + X_* = 0, \quad \nabla_\alpha {}^*L_*^\alpha + \rho_\alpha{}^* \times f_*^\alpha + M_* = 0 \qquad (2.5)$$

and the static boundary conditions

$$\Phi_* = P - \frac{\partial m_* H}{\partial s^*}, \quad G_* = M^{\alpha\beta} n_\alpha {}^*n_\beta \qquad (2.6)$$

where

$$P = (T_*^{\alpha\beta}\rho_\beta{}^* + N_*^\alpha m_*)n_\alpha{}^*, \qquad H = -M^{\alpha\beta} n_\alpha{}^* \tau_\beta{}^* \qquad (2.7)$$

$$f_*^\alpha = T_*^{\alpha\beta}\rho_\beta{}^* + N_*^\alpha m^*, \qquad L_*^\alpha = (m_* \times \rho_\beta{}^*)M^{\alpha\beta} \qquad (2.8)$$

Putting $\Phi^* = \Phi_*^\alpha \rho_\alpha + \Phi_*^3 m_*$, from (2.6) we obtain the scalar form

$$\Phi_*^\alpha = T_*^{\beta\alpha} n_\beta{}^* + b^*{}_\beta^\alpha \tau_\beta{}^* H, \qquad \Phi_3{}^* = N_*^\alpha n_\alpha{}^* - \frac{\partial H}{\partial s^*} \qquad (2.9)$$

If stretching and shear are negligible with respect to unity, the relation (2.2) can be written in the form

$$\delta A = \iint_{\sigma^*} (T^{\alpha\beta}\,\delta p_{\alpha\beta} + M^{\alpha\beta}\,\delta\mu_{\alpha\beta})\,d\sigma^* \tag{2.10}$$

where

$$\mu_{\alpha\beta} = -q_{\alpha\beta} + {}^*b_\alpha^\gamma p_{\gamma\beta} \tag{2.11}$$

can be taken as the second strain tensor[3, 4].

Let $T^{\alpha\beta}$ be the tangential stresses in pure strain axes[4], and $\varepsilon_{\alpha\beta}$ the components of the pure strain:

$$2p_{\alpha\beta} = 2\varepsilon_{\alpha\beta} + \varepsilon_\alpha^\gamma \varepsilon_{\gamma\beta} \tag{2.12}$$

Then putting $\delta p_{\alpha\beta}$ from (2.12) into (2.2) we find δW, the increment in the strain energy of the shell referred to a unit area of the unstrained median surface:

$$\delta W = T_*^{\alpha\beta}\,\delta p_{\alpha\beta} + {}^*b_\gamma^\alpha M^{\gamma\beta}\,\delta p_{\alpha\beta} - M^{\alpha\beta}\,\delta q_{\alpha\beta} =$$

$$T^{\alpha\beta}\delta\varepsilon_{\alpha\beta} + \sqrt{\frac{a^*}{a}}\,({}^*b_\gamma^\alpha M^{\gamma\beta}\,\delta\varepsilon_{\alpha\beta} - M^{\alpha\beta}\,\delta q_{\alpha\beta})$$

where

$$T^{\alpha\beta} = \sqrt{\frac{a^*}{a}}\,[T_*^{\alpha\beta} + \frac{1}{2}T_*^{\alpha\gamma}\varepsilon_\gamma^\beta + \frac{1}{2}T_*^{\beta\gamma}\varepsilon_\gamma^\alpha +$$

$$\frac{1}{2}{}^*b_\gamma^\alpha(M^{\gamma\sigma}\varepsilon_\sigma^\beta + M^{\beta\sigma}\varepsilon_\sigma^\gamma)]$$

As a result, for small strains $T^{\alpha\beta} \approx T_*^{\alpha\beta}$, $p_{\alpha\beta} \approx \varepsilon_{\alpha\beta}$. A theory of shells, in which the magnitudes are related to the axes "after rotation" (to the axes of pure strain) is built up in the work of N. A. Alumyae[4].

The basic equations of the nonlinear theory of shells in Euler variables and their transformation to Lagrange variables are given in articles by Synge and Chien[5, 6]. However the accuracy of the equations obtained is greater than the accuracy of a linear Hooke's law. Therefore the theory given in these works becomes comprehensible only as a result of great simplifications[7]. In the derivation of the equilibrium equations (2.4), the coordinate system is made up of Gaussian coordinates x^1 and x^2 on S^* and a third coordinate z, which is the distance of a point on the deformed shell to the surface S^*.

If use is made of the relations (1.10), the equilibrium equations are transformed to the form

$$\nabla_\alpha \left(\sqrt{\frac{a_*}{a}}\, T_*^{\alpha\beta}\right) + \sqrt{\frac{a_*}{a}}\left(a_*^{\beta\lambda} p_{\lambda,\alpha\gamma} T_*^{\alpha\gamma} - {}^*b_\alpha^\beta N_*^\alpha + X_*^\beta\right) = 0 \tag{2.13}$$

$$\nabla_\alpha \left(\sqrt{\frac{a_*}{a}}\, N_*^\alpha\right) + \sqrt{\frac{a_*}{a}}\left(b_{\alpha\beta}\, {}^*T_*^{\alpha\beta} + X_*^3\right) = 0 \tag{2.14}$$

$$\nabla_\alpha \left(\sqrt{\frac{a_*}{a}}\, H^{\alpha\beta}\right) + \sqrt{\frac{a_*}{a}}\left(a_*^{\beta\lambda} p_{\lambda,\alpha\gamma} H^{\alpha\gamma} - N_*^\beta + M_*^\alpha a_*^{\lambda\beta} c_{\alpha\lambda}^*\right) = 0 \tag{2.15}$$

and in the case of small strains ($a_* \approx a$, $a_*^{\alpha\beta} \approx a^{\alpha\beta}$) to the form

$$\nabla_\alpha T_*^{\alpha\beta} + a^{\beta\lambda} p_{\lambda,\alpha\gamma} T_*^{\alpha\gamma} - a^{\beta\gamma}(b_{\alpha\gamma} + q_{\alpha\gamma}) N_*^\alpha + X^\beta = 0 \tag{2.16}$$

$$\nabla_\alpha N_*^\alpha + (b_{\alpha\beta} + q_{\alpha\beta}) T_*^{\alpha\beta} + X^3 = 0 \tag{2.17}$$

$$\nabla_\alpha H^{\alpha\beta} - N_*^\beta + a^{\beta\lambda} p_{\lambda,\alpha\gamma} H^{\alpha\gamma} + M^\alpha a^{\lambda\beta} c_{\alpha\lambda} = 0 \tag{2.18}$$

Here X^β, X^3, M^α are the components of the external forces and moments in the coordinate system of the deformed shell related to a unit area of the undeformed surface. The boundary conditions (2.9) after substitution of the covariant components of the vectors normal and tangent to the undeformed contour of the shell can be written in the form

$$\Phi_* \frac{ds^*}{ds} = n_\alpha T_*^{\alpha\beta} + \tau^\beta b^{*\alpha}_\beta H, \quad \Phi_3 \frac{ds^*}{ds} = n_\alpha N_*^\alpha - \frac{\partial H}{\partial s} \tag{2.19}$$

$$\sqrt{\frac{a}{a_*}}\left(\frac{ds_*}{ds}\right)^2 G_* = H^{\alpha\beta} n_\alpha n_\beta$$

where n_α and τ_α are the components normal and tangent to the contour C in a system of coordinates on the surface S, and ds and ds* are elements of path length of the contour of the shell up to and after deformation:

$$ds^* = ds\sqrt{1 + 2c^{\alpha\pi} c^{\beta\lambda} p_{\alpha\beta} n_\alpha n_\beta}$$

It should be noted that (2.16), (2.17) and (2.18) contain in an unexplicit form the equilibrium equations of the theory of shells for large displacements obtained by Lyav and other authors. If one retains in them second-order infinitesimals with respect to displacements and their derivatives, i.e., if one puts

$$P_{\gamma,\alpha\beta} = b_{\alpha\beta}\,\omega_\gamma - b_{\alpha\gamma}\,\omega_\beta + \nabla_\alpha e_{\beta\gamma}, \quad q_{\alpha\beta} = B_{\alpha\beta} \tag{2.20}$$

then they go over into the equations of Lyav in generalized coordinates.

We suppose that $q_{\alpha\beta} \sim \sqrt{\varepsilon_p}$, $e_{\alpha\beta} \sim \varepsilon_p$, where the symbol $\sim$ indicates that the quantities compared have the same order of magnitude, and ε_p is the maximum relative extension within the limits of proportionality. The theory of shells based on these assumptions is comparable in accuracy with von Karman's theory of plates. We call this kind of a deflection of a shell a mean deflection. In this case the following approximate formulas are correct:

$$2p_{\alpha\beta} = e_{\alpha\beta} + e_{\beta\alpha} + \omega_\alpha \omega_\beta, \quad q_{\alpha\beta} = B_{\alpha\beta} + b_\alpha^\gamma \omega_\gamma \omega_\beta \tag{2.21}$$

$$P_{\gamma,\alpha\beta} = \omega_\gamma (b_{\alpha\beta} + B_{\alpha\beta}) + \Omega_{\alpha\beta\gamma}$$

Substituting $q_{\alpha\beta}$ and $P_{\gamma,\alpha\beta}$ from here into (2.16), (2.17) and (2.18) we get an equation for the mean deflection; referred to the lines of curvature they agree with the equilibrium equations of Kh. M. Mushtari[8]. These refined equations are necessary in problems of stability and in impact phenomena.

If the external forces and moments do not depend on the strains, the problems in the theory of shells can be solved either in terms of forces and moments or in components of the strains, by adding to the system (2.16), (2.17), and (2.18) the common strain conditions.

The equilibrium equations for the theory of small strains (2.16), (2.17) and (2.18) can be simplified. Up to the loss of stability, terms containing small multiples of $P_{\gamma,\alpha\beta}$ can be neglected since they are quantities of the order $h\varepsilon_p^2$ and in the presence of edge effects the ratio of these terms to the main ones will be of the order ε_p. The neglect of these quadratic terms amounts to a significant limitation only when terms of the equations, which we have assumed to be the principal ones, themselves are small on account of the mutual cancellation of their principal terms (for instance, in the solution of the problem of the stability of very long cylindrical tubes during axial compression.

3. <u>Introduction of Symmetrical Tensors of Force and Momentum.</u> We introduce symmetric tensors of the tangential forces and moments, putting

$$H^{\alpha\beta} = M_*^{\alpha\beta} + Q_*^{\alpha\beta}, \quad T_*^{\alpha\beta} = s_*^{\alpha\beta} - {}^*b_\gamma^\beta H^{\gamma\alpha} \tag{3.1}$$

where

$$2M_*^{\alpha\beta} = H^{\alpha\beta} + H^{\beta\alpha}, \quad 2Q_*^{\alpha\beta} = H^{\alpha\beta} - H^{\beta\alpha}$$

Substituting $T_*^{\alpha\beta}$ in the sixth equilibrium equation

$$c_{\alpha\beta} {}^*T_*^{\alpha\beta} + {}^*b_\gamma^\beta c_{\alpha\beta} {}^*H^{\gamma\alpha} = 0 \tag{3.2}$$

we get $c_{\alpha\beta}\, s_*^{\alpha\beta} = 0$, i.e., $s_*^{\alpha\beta}$ is a symmetric tensor.

An antisymmetric tensor $Q^{\alpha\beta} = -Q^{\beta\alpha}$ can be determined from the supplementary nondifferential condition which assures matching of the tangential tensions at all points through the thickness of the shell.

On retaining the term of smallest degree in h this relation will be

$$h^2 c_{\gamma\alpha} {}^*b^*{}_\beta^\gamma T_*^{\alpha\beta} - 3c_{\alpha\beta} {}^*H^{\alpha\beta} = 0 \tag{3.3}$$

Putting (3.1) in (2.2) and considering that the scalar product of a symmetric tensor by an antisymmetric one is equal to zero, we obtain

$$\delta A = \iint_{\sigma^*} (s_*^{\alpha\beta}\, \delta p_{\alpha\beta} - M_*^{\alpha\beta}\, \delta q_{\alpha\beta})\, d\sigma^* = \tag{3.4}$$

$$\iint_{\sigma^*} \Big\{ (s_*^{\alpha\beta} - {}^*b_\gamma^\beta M^{\gamma\alpha})\, \rho_\beta^* \cdot \delta\rho_\alpha^* + M_*^{\alpha\beta} \rho_\beta^* \cdot \frac{\partial \delta m_*}{\partial x^\alpha} \Big\} d\sigma^*$$

From this the equilibrium equations follow*

*Note: It follows from (3.5) that the simplification $T_*^{\alpha\beta} = T_*^{\beta\alpha}$ in the case of a thin shell is only permissible when the shear forces N_*^α of the first two equilibrium equations are <u>negligible.</u>

$$\nabla_\alpha {}^*(s_*^{\alpha\beta} - {}^*b_\gamma^\beta M_*^{\gamma\alpha}) - {}^*b_\alpha^\beta Q_*^\alpha + X_*^\beta = 0 \tag{3.5}$$

$$\nabla_\alpha {}^*Q_*{}^\alpha + b_{\alpha\beta}\, {}^*(s_*{}^{\alpha\beta} - {}^*b_\gamma^\beta M_*{}^{\gamma\alpha}) + X_*{}^3 = 0 \qquad (3.6)$$

$$\nabla_\alpha {}^*M_*{}^{\alpha\beta} - Q_*{}^\beta - s_*{}^{\gamma\beta} c_{\alpha\gamma}\, {}^*M_*{}^\alpha = 0 \qquad (3.7)$$

and the static boundary conditions

$$\Phi_*{}^\alpha = (s_*{}^{\alpha\beta} - {}^*b_\gamma^\beta M_*{}^{\gamma\alpha}) n_\beta{}^* + {}^*b_\beta^\alpha \tau_*{}^\beta H_1 \qquad (3.8)$$

$$\Phi_3{}^\sigma = Q_*{}^\alpha n_\alpha{}^* - \frac{\partial H_1}{\partial s^*},\quad G = M_*{}^{\alpha\beta} n_\alpha{}^* n_\beta{}^*$$

where $H_1 = -M_*{}^{\alpha\beta} n_\alpha{}^* \tau_\beta{}^*$ is the twisting moment on the contour, $Q_*{}^\alpha$ is a vector similar to the vector of the shear forces. Thus there are essentially six unknowns $s_*{}^{\alpha\beta}$ and $M_*{}^{\alpha\beta}$ in the theory of shells instead of eight. The same observations were made by A. I. Lur'ye[9]. Since for an isothermal deformation process the work of deformation is a total differential, from (3.4) we have the general relations of elasticity

$$\sqrt{\frac{a_*}{a}}\, s_* = \frac{\partial W}{\partial p_{\alpha\beta}},\quad \sqrt{\frac{a_*}{a}}\, M_*{}^{\alpha\beta} = -\frac{\partial W}{\partial q_{\alpha\beta}} \qquad (3.9)$$

where W is the energy of deformation of the shell, related to a unit area of the mean undeformed surface.

We introduce still another tensor of forces and moments, putting

$$\sqrt{\frac{a_*}{a}}\, s_* - \ = s^{\alpha\beta} = \frac{\partial W}{\partial p_{\alpha\beta}},\quad \sqrt{\frac{a_*}{a}}\, M_*{}^{\alpha\beta} = M^{\alpha\beta} = -\frac{\partial W}{\partial q_{\alpha\beta}} \qquad (3.10)$$

$$\sqrt{\frac{a_*}{a}}\, Q_*{}^\alpha = Q^\alpha$$

as well as vectors of the external forces and moments, related to a unit surface of the mean undeformed surface:

$$X = \sqrt{\frac{a_*}{a}}\, X_*,\quad M = \sqrt{\frac{a_*}{a}}\, M_* \qquad (3.11)$$

Then from (3.5)-(3.7) we get the equations of equilibrium in the form

$$\nabla_\alpha (s^{\alpha\beta} - {}^*b_\gamma^\beta M^{\gamma\alpha}) + a_*{}^{\beta\gamma} P_{\lambda,\alpha\sigma}(s^{\gamma\sigma} - {}^*b_\gamma^\sigma M^{\gamma\sigma}) - \qquad (3.12)$$
$${}^*b_\alpha^\beta Q^\alpha + X^\beta = 0$$

$$\nabla_\alpha Q^\alpha + b^*{}_{\alpha\beta} (s^{\alpha\beta} - {}^*b_\gamma{}^\beta M^{\gamma\alpha}) + X^3 = 0 \qquad (3.13)$$

$$\nabla_\alpha M^{\alpha\beta} + a_*^{\beta\lambda} p_{\lambda,\alpha\gamma} M^{\alpha\gamma} - Q^\beta - M^\alpha a_*^{\gamma\beta} c_{\alpha\gamma} = 0 \qquad (3.14)$$

and from (3.8) the static boundary conditions in the form

$$\Phi^\alpha = (s^{\alpha\beta} - b^{*\beta}_{\gamma} M^{\gamma\alpha}) n_\beta + b^{*\alpha}_{\beta} \tau^\beta H_2, \quad \Phi_3 = Q^\alpha n_\alpha - \frac{\partial H_2}{\partial s} \qquad (3.15)$$

$$\sqrt{\frac{a}{a_*}}\frac{ds^*}{ds} G = M^{\alpha\beta} n_\alpha n_\beta, \quad \left(\frac{ds^*}{ds}\right)^2 H_2 = -M^{\alpha\beta} n_\alpha \tau_\beta$$

In these equations ∇ is the sign of covariant differentiation with respect to $a_{\alpha\beta}$; X^α, M^α are the contravariant components of the vectors X and M in the coordinate system of the mean deformed surface; $X^3 = n_* X$; Φ^α and Φ_3 are the components of the contour force related to a unit length of the undeformed contour C of the shell in the same system of coordinates; G is the bending moment related to a unit length of the undeformed contour. It is important to note that for usual displacements and strains the first and second strain tensors admit of a potential. In fact, introducing the function F defined by

$$F = s^{\alpha\beta} p_{\alpha\beta} - M^{\alpha\beta} q_{\alpha\beta} - W \qquad (3.16)$$

from the relation

$$\delta F = \frac{\partial F}{\partial s^{\alpha\beta}} \delta s^{\alpha\beta} + \frac{\partial F}{\partial M^{\alpha\beta}} \delta M^{\alpha\beta} = p_{\alpha\beta} \delta s^{\alpha\beta} - q_{\alpha\beta} \delta M^{\alpha\beta} + s^{\alpha\beta} \delta p_{\alpha\beta} - M^{\alpha\beta} \delta q_{\alpha\beta} - \delta W$$

taking (3.10) into account we obtain formulas similar to the Castigliano formulas in the linear theory:

$$p_{\alpha\beta} = \frac{\partial F}{\partial s^{\alpha\beta}}, \quad p_{\alpha\beta} = -\frac{\partial F}{\partial M^{\alpha\beta}} \qquad (3.17)$$

In the case of small displacements and shears we have

$$2F = s^{\alpha\beta} p_{\alpha\beta} - M^{\alpha\beta} q_{\alpha\beta} \qquad (3.18)$$

In this case for an elastic condition of the shell, F is the work of deformation, and for an elastic-plastic condition it is the additional work. We present equations (3.5)-(3.7) in the vectorial form

$$\nabla_\alpha {}^*f_1^\alpha + X_* = 0, \quad \nabla_\alpha {}^*L_*^\alpha + \rho_\alpha{}^* \times f_1^\alpha + M_* = 0 \qquad (3.19)$$

where

$$f_1^{\alpha} = s_*^{\alpha\beta} \rho_\beta^* + M_*^{\alpha\beta} m_\beta^* + m_* Q_*^{\alpha}, \quad L_*^{\alpha} = (m_* \times \rho_\beta^*) M_*^{\alpha\beta} \tag{3.20}$$

then the equations are satisfied for $X_* = M_* = 0$, putting

$$f_1^{\alpha} = c_*^{\alpha\beta} \nabla_\beta^* \varphi, \quad L_*^{\alpha} = c_*^{\alpha\beta} [\nabla_\beta^* \psi + (\rho_\beta^* \times \varphi)] \tag{3.21}$$

where φ and ψ are certain vectors. Since $L_*^{\alpha} \cdot m_* = 0$, these vectors satisfy the conditions

$$\varphi^{\alpha} = c_*^{\alpha\beta} (\nabla_\beta \psi + \psi^{\lambda} b_{\lambda\beta}^*) \tag{3.22}$$

and

$$\varphi = \varphi^{\alpha} \rho_\alpha^* + m_* \varphi, \quad \psi = \psi^{\alpha} \rho_\alpha^* + m_* \psi \tag{3.23}$$

From a comparison of (3.20) and (3.21) we get

$$s_*^{\alpha\beta} = c_*^{\alpha\pi} c_*^{\beta\gamma} \nabla_\pi^* (\nabla_\gamma \psi + b^{*\sigma}_{\gamma} \psi) - \tag{3.24}$$

$$^*b^{\beta}_{\gamma} c_*^{\alpha\sigma} c_*^{\gamma\pi} (\nabla_\pi^* \psi_\sigma - b_{\pi\sigma}^* \psi)$$

$$M_*^{\alpha\beta} = c_*^{\alpha\pi} c_*^{\beta\gamma} (\nabla_\pi^* \psi_\gamma - b_{\pi\gamma}^* \psi) - c_*^{\alpha\beta} \varphi \tag{3.25}$$

$$Q_*^{\alpha} = c_*^{\alpha\beta} c_*^{\mu\nu} b_{\beta\nu}^* (b^{*\lambda}_{\nu} \psi_\lambda + \nabla_\nu \psi) + c_*^{\alpha\beta} \nabla_\beta^* \varphi \tag{3.26}$$

Since $M_*^{\alpha\beta} c_{\alpha\beta}^* = 0$, it follows from (3.25) that $2\varphi = c_*^{\alpha\beta} \nabla_\alpha^* \psi_\beta$. Thus the general solution of the homogeneous system contains three independent functions ψ and ψ_α.

4. <u>Laws of Elasticity and Strengthening</u>. Let ε_{ik} be the covariant components of the finite strain tensor ($\varepsilon^i_k = g^{is} \varepsilon_{ks}$), σ^{ik} the contravariant components of the stress tensor in the coordinate system of the shell after deformation; and W^o the energy density for deformation of the shell considered as a three-dimensional body. Then for an isotropic shell during deformations of usual magnitude we have the nonlinear relations

$$\sigma^{ik}\sqrt{g} = g^{ik}\left(\frac{\partial W^o}{\partial S_1} + S_1 \frac{\partial W^o}{\partial S_2} + S_2 \frac{\partial W^o}{\partial S_3}\right) - \left(\frac{\partial W^o}{\partial S_2} + S_1 \frac{\partial W^o}{\partial S_3}\right)\varepsilon^{ik} +$$

$$\frac{\partial W^o}{\partial S_3}\varepsilon^{ij}\varepsilon^k_j$$

where W^o depends only on the invariants of the tensor ε^{i}_{k}.

$$S_1 = g^{ik}\varepsilon_{ik},\quad 2S_2 = S_1^2 - \varepsilon^i_k\varepsilon^k_i,\quad S_3 = \det(\varepsilon^i_k), \tag{4.1}$$

$$S = 1 + 2S_1 + 4S_2 + 8S_3$$

If we take the Kirchhof hypothesis ($\varepsilon_3^1 = \varepsilon_3^2 = 0$), then the relations of (4.1) are considerably simplified, since for this case

$$S_1 = \Theta_1 + \varepsilon_3^3,\quad S_2 = \Theta_1\varepsilon_3^3 + \Theta_2,\quad S_3 = \Theta_2\varepsilon_3^3 \tag{4.2}$$

where the quantities Θ_1 and Θ_2 are expressed by the formulas

$$\Theta_1 = \frac{a}{g} c^{\alpha\beta} c^{\pi\lambda} g_{\beta\lambda}\varepsilon_{\alpha\pi},\quad \Theta_2 = \frac{a}{2g} c^{\alpha\beta} c^{\pi\lambda}\varepsilon_{\alpha\pi}\varepsilon_{\beta\lambda} \tag{4.3}$$

and are the invariants with respect to a transformation of Gaussian coordinates on the surfaces $x^3 = \text{const.}$ To determine ε_3^3 we take as usual $\sigma^{33} = 0$. Then putting in (4.1) $i = k = 3$ and introducing in the relations obtained S_1, S_2, and S_3 from (4.2), we obtain an equation of the form

$$f(\Theta_1, \Theta_2, \varepsilon_3^2) = 0 \tag{4.4}$$

We note further that in the relations (4.1) for $i, k = 1, 2$ the expressions

$$(\varepsilon^{ij}\varepsilon^k_j - S_1\varepsilon^{ik})\frac{\partial W^0}{\partial S_3} = -(\varepsilon_3^3\varepsilon^{\alpha\beta} + g^{\alpha\beta}\Theta_2)\frac{\partial W^0}{\partial S_3}$$

Then substituting this last expression in (4.1) and taking into account (4.2) and (4.4) we get the condition for the elasticity of an isotropic shell

$$\sigma^{\alpha\beta} = A_1^{\alpha\beta} + B_1^{\alpha\beta\pi\lambda}\varepsilon_{\pi\lambda}\quad (\alpha, \beta, \pi, \lambda = 1.2) \tag{4.5}$$

where $A_1^{\alpha\beta}$ and $B_1^{\alpha\beta\pi\lambda}$ are functions of the invariants Θ_1 and Θ_2 only.

In order to express the forces and moments by means of the deformations of the surface it is necessary to consider that Θ_1 and Θ_2 depend on the following invariants of the strain tensors of the surface:

$$p_1 = a^{\alpha\beta}p_{\alpha\beta},\quad q_1 = a^{\alpha\beta}q_{\alpha\beta},\quad p_2 = \det(p^\alpha_\gamma),\quad q_2 = \det(q^\alpha_\gamma) \tag{4.6}$$

$$p_1' = b^{\alpha\beta}p_{\alpha\beta},\quad q_1' = b^{\alpha\beta}q_{\alpha\beta},\quad r = c^{\alpha\beta}c_{\pi\lambda}q^\pi_\alpha p^\lambda_\beta,\quad r' = p^\alpha_\gamma q^\gamma_\alpha$$

Then W, the potential of the forces and moments, will depend in the general case on these eight parameters.

Therefore, we note that

$$\frac{\partial p_1}{\partial p_{\alpha\beta}} = a^{\alpha\beta}, \quad \frac{\partial p_1'}{\partial p_{\alpha\beta}} = b^{\alpha\beta}, \quad \frac{\partial p_2}{\partial p_{\alpha\beta}} = \frac{\partial r}{\partial q_{\alpha\beta}} = \tag{4.7}$$

$$a^{\alpha\beta} p_1 - p^{\alpha\beta}, \quad \frac{\partial r'}{\partial p_{\alpha\beta}} = q^{\alpha\beta}, \quad \frac{\partial r_1}{\partial q_{\alpha\beta}} = p^{\alpha\beta},$$

$$\frac{\partial q_2}{\partial q_{\alpha\beta}} = \frac{\partial r}{\partial p_{\alpha\beta}} = a^{\alpha\beta} q_1 - q^{\alpha\beta}$$

From (3.10) we obtain the condition of elasticity

$$s^{\alpha\beta} = \frac{\partial W}{\partial p_{\alpha\beta}} = D_1^{\alpha\beta} + D_2 p^{\alpha\beta} + D_3 q^{\alpha\beta}, \quad M^{\alpha\beta} = -\frac{\partial W}{\partial q_{\alpha\beta}} = \tag{4.8}$$

$$D_4^{\alpha\beta} + D_3 p^{\alpha\beta} + D_5 q^{\alpha\beta}$$

where

$$D_1^{\alpha\beta} = a^{\alpha\beta}\left(\frac{\partial W}{\partial p_1} + p_1 \frac{\partial W}{\partial p_2} + q_1 \frac{\partial W}{\partial r}\right) + b^{\alpha\beta} \frac{\partial W}{\partial p_1'}, \tag{4.9}$$

$$D_3 = \frac{\partial W}{\partial r'} - \frac{\partial W}{\partial r}, \quad D_4^{\alpha\beta} = a^{\alpha\beta}\left(\frac{\partial W}{\partial q_1} + p_1 \frac{\partial W}{\partial r} + q_1 \frac{\partial W}{\partial q_2}\right) +$$

$$b^{\alpha\beta} \frac{\partial W}{\partial q_1'}, \quad D_2 = -\frac{\partial W}{\partial p_2}, \quad D_5 = -\frac{\partial W}{\partial q_2}$$

Relation (4.8) is applicable to deformations of usual magnitude and is independent of the Kirchhof hypothesis for conditions such that

$$W = \int_{-h}^{h} W^0 dx^3$$

We consider the case of small elastic-plastic deformations of a shell with large displacements. In this case to the limits of accuracy of the Kirchhof hypothesis the formulas of the first approximation of Kirchhof-Lyaw may be used. The work of volume deformations in the plastic state is

$$W_0 = K\left(\frac{2\mu}{\lambda + 2\mu}\right)^2 \left(hp_1{}^2 + \frac{h^3}{3} q_1{}^2\right) \tag{4.10}$$

where 2h is the thickness of the shell, λ, μ are Lame constants, and K is the bulk compression modulus. Since the energy density for change of form in the plastic state of a material depends only on the second invariant of the deviation of the deformation, the invariants p_2, q_2, r, r' enter the expression W through the invariants

$$P_p = (p_1)^2 - p_2,\ P_q = (q_1)^2 - q_2,\ P_{pq} = r' + \tfrac{1}{2} r \tag{4.11}$$

In view of this and of the relations (4.7) and (4.10) we have

$$S^{\alpha\beta} = A^{\alpha\beta\pi\lambda} p_{\pi\lambda} + B^{\alpha\beta\pi\lambda} q_{\pi\lambda},\ M^{\alpha\beta} = B^{\alpha\beta\pi\lambda} p_{\pi\lambda} + C^{\alpha\beta\pi\lambda} q_{\pi\lambda} \tag{4.12}$$

where

$$A^{\alpha\beta\pi\lambda} = a^{\alpha\beta} a^{\pi\lambda} \left[2hK\left(\frac{2\mu}{\lambda + 2\mu}\right)^2 + \frac{2}{3} I_1\right] + \frac{2}{3} I_1 a^{\alpha\pi} a^{\beta\lambda} \tag{4.13}$$

$$B^{\alpha\beta\pi\lambda} = -\frac{2}{3} I_2 \left(a^{\alpha\beta} a^{\pi\lambda} + a^{\alpha\pi} a^{\beta\lambda}\right)$$

$$C^{\alpha\beta\pi\lambda} = \left[\frac{2h}{3} K\left(\frac{2\mu}{\lambda + 2\mu}\right)^2 + \frac{2}{3} I_3\right] a^{\alpha\beta} a^{\pi\lambda} + \frac{2}{3} I_3 a^{\alpha\pi} a^{\beta\lambda}$$

where

$$I_1 = \frac{3}{2} \frac{\partial W}{\partial P_p},\ I_2 = -\frac{3}{4} \frac{\partial W}{\partial P_{pq}},\ I_3 = \frac{3}{2} \frac{\partial W}{\partial P_q} \tag{4.14}$$

are integrals introduced by A. A. Ilyushin[10] in the case of plates.

5. <u>The Lagrange-Galerkin Equations</u>. Finding an exact solution of the equations of the theory of shells during large displacements is difficult even in the simplest cases. Therefore, application is made of approximate methods based on variational theorems. In problems of the stability of elastic equilibrium and the investigation of impulse phenomena in shells, the variational equation (2.2) is usually used, expressing the origin of the possible displacements

$$\delta A = \iint_{\sigma^*} (T_*^{\alpha\beta} \delta p_{\alpha\beta} + {}^*b_\gamma^\alpha H^{\gamma\beta} \delta p_{\alpha\beta} - \qquad (5.1)$$

$$H^{\alpha\beta} \delta q_{\alpha\beta}) d\sigma^* = \iint_{\sigma^*} (s_*^{\alpha\beta} \delta p_{\alpha\beta} - M_*^{\alpha\beta} \delta q_{\alpha\beta}) d\sigma^*$$

If the external forces permit a potential, then the discovery of functions satisfying equation (5.1) comes down to finding a minimum of the total potential energy (Ritz method). The approximation method of Ritz can only be applied to the case where the external forces are given independently of the deformation or they are of the form of a hydrostatic pressure. If the external forces do not satisfy such conditions the problem of finding the critical loading can be solved by application of the variational equations (5.1) or by the Galerkin method.

We note that while in the linear theory of shells the Ritz equations and the Galerkin equations result from the cause of the possible displacements and are identities, in the nonlinear theory the Galerkin equations have no kind of connection with the energy function. The Galerkin equations in the nonlinear theory of shells are derived not with reference to the cause of the possible displacements (5.1) but from the variational equations:

$$\iint_{\sigma^*} (X_*^\alpha \delta v_\alpha + X_*^3 \delta w + M_*^\beta a_*^{\alpha\lambda} c_{\beta\lambda}{}^* \delta\tilde{\omega}_\alpha) d\sigma^* + \qquad (5.2)$$

$$\int_{C^*} (\Phi_*^\alpha \delta v_\alpha + \Phi_*^3 \delta w - G_* n_*^\alpha \delta\tilde{\omega}_\alpha) ds^* =$$

$$\iint_{\sigma^*} \{ T_*^{\alpha\beta} (\nabla_\alpha{}^* \delta v_\beta - b_{\alpha\beta}{}^* \delta w) - H^{\alpha\beta} \nabla_\alpha{}^* \delta\tilde{\omega}_\beta \} d\sigma^*$$

where

$$\delta\tilde{\omega}_\alpha = \nabla_\alpha \delta w + {}^*b_\alpha^\lambda \delta v_\lambda$$

In fact, integrating by parts we obtain from (5.2)

$$\int_{C^*} \{ (\Phi - F + \frac{\partial\, m^* H}{\partial s^*})(\rho_*^\alpha \delta v_\alpha + m_* \delta w) + (c_*^{\alpha\beta} G_* \tau_\beta{}^* - \qquad (5.3)$$

$$H^{\beta\alpha} n_\beta{}^*) \delta\tilde{\omega}_\alpha \} ds^* = \iint_{\sigma^*} (\nabla_\alpha {}^*T_*^{\alpha\beta} - {}^*b_\alpha^\beta H_*^\alpha +$$

$$X_*^\beta) \delta v_\alpha + (\nabla_\alpha {}^*H_*^\alpha + b_{\alpha\beta} {}^*T_*^{\alpha\beta} + X_*^3) \delta w +$$

$$(\nabla_\alpha {}^*H^{\alpha\beta} - N_*^\beta - M_*^\alpha a_*^{\gamma\beta} c_{\alpha\gamma}{}^*) \delta\tilde{\omega}_\alpha \} d\sigma^*$$

Hence, substituting

$$\delta v_\alpha = \sum f_{\alpha\gamma}\,\delta A^\lambda, \quad \delta w = \sum f_\lambda\,\delta B^\lambda$$

where A^λ and B^λ are constants, we get the Galerkin equations.

As is known, when the same system of functions $f_{\alpha\lambda}$ and f_λ are chosen the results of the solution of the linear problems by the method of Ritz and Galerkin is identical; the Galerkin method is used to reduce significantly the amount of computational work. This to a large degree is the advantage of the Galerkin method in nonlinear problems also. This can be seen just from a comparison of the variational equations (5.1) and (5.2), from which the Ritz and Galerkin equations follow, and also from a comparison of the relations (5.1) and (5.3). The right-hand side of (5.3) is obtained from the right-hand side of (5.1) by calculating the expression

$$\iint_{\sigma^*} \{(\nabla_\alpha {}^*T_*^{\alpha\beta} - {}^*b_\alpha^\beta N_*^\alpha + X_*^\beta)\,R_\beta + (\nabla_\alpha {}^*N_*^\alpha +$$

$$b_{\alpha\beta}\,{}^*T_*^{\alpha\beta}\,X_*^3)\,R_3 + (\nabla_\alpha {}^*H^{\alpha\beta} - N_*^\beta - M_*^\alpha a_*^{\gamma\beta} c_{\alpha\gamma}{}^*)\,\Omega_\alpha\}\,d\sigma^*$$

where

$$R_\beta = e_\alpha^\lambda\,\delta v_\lambda + \omega_\alpha\,\delta w, \quad R_3 = \sqrt{\frac{a}{a_*}}\,(E_0\,\delta w + E^\alpha\,\delta v_\alpha) - \delta w,$$

$$\Omega_\alpha = -\delta\omega_\alpha - \rho_\alpha{}^* \cdot \delta m_*$$

Together with (5.1) the relation (5.2) can be successfully applied to the solution of problems in the nonlinear theory. For small strains this relation is simplified, since in a number of cases it is possible to neglect differences in covariant differentiation with respect to $a_{\alpha\beta}$ and $a_{\alpha\beta}{}^*$. In addition, it is possible to neglect the affect of forces of the type $N_*^\alpha q_\alpha^\beta$ on the possible displacements δv_β. Thus instead of (5.2) we get

$$\iint_\sigma (X_*^\alpha\,\delta v_\alpha + X_*^3\,\delta w + M_*^\beta a^{\lambda\alpha} c_{\beta\lambda}\,\delta\omega_\alpha)\,d\sigma + \tag{5.4}$$

$$\int_C (\Phi_*^\alpha\,\delta v_\alpha + \Phi_*^3\,\delta w - G_*n^\alpha\,\delta\omega_\alpha)\,ds = \iint_\sigma (T_*^{\alpha\beta}\,\delta e_{\alpha\beta} -$$

$$T_*^{\alpha\beta} q_{\alpha\beta}\,\delta w - H^{\alpha\beta}\,\delta\nabla_\alpha\omega_\beta)\,d\sigma$$

The right-hand side of this relation is only slightly different in form from the strain energy calculated according to the linear theory, but the external forces and moments are given in

the coordinate system of the deformed shell. Let X^{α} and X^3 be the components of the external forces, and $M_{1\alpha}$ the components of the external momentum in the coordinate system of the undeformed shell. According to the formulas given in our paper[11]:

$$X_*^{\alpha} = X_{\beta}(a^{\alpha\beta} + e^{\alpha\beta}) + X^3\omega^{\alpha},\ X_*^3 = E_0X^3 + E_{\alpha}X^{\alpha} \tag{5.5}$$

$$M_*^{\beta} = c^{\pi\alpha}\left\{M_{1\alpha}c^{\lambda\beta}a_{\pi\nu}(\delta^{\nu}_{\lambda} + e^{\nu}_{\lambda}) + \delta^{\beta}_{\pi}\omega_{\alpha}Y\right\},$$

$$Y = \bar{F}(+) - p(-)h$$

$$\Phi_*^{\alpha} = \Phi_{\alpha}(a^{\alpha\beta} + e^{\alpha\beta}) + \Phi_3\omega^{\alpha},\ \Phi^3 = E_0\Phi^3 + E_{\alpha}\Phi^{\alpha} \tag{5.6}$$

where Φ_{α} and Φ_3 are the components of the contour force in the system of coordinates of the undeformed shell. If one neglects the quantities $e_{\alpha\beta}$ with respect to unity, from (5.5) and (5.6) we have

$$X_*^{\alpha} = X^{\alpha} + X^3\omega^{\alpha},\ X_*^3 = X^3 - X^{\alpha}\omega_{\alpha},\ M_*^{\beta} = M_1^{\beta}$$

$$\Phi_*^{\alpha} = \Phi^{\alpha} + \Phi_3\omega^{\alpha},\ \Phi_*^3 = \Phi^3 - \Phi^{\alpha}\omega_{\alpha}$$

In addition, it is sufficiently accurate to put $q_{\alpha\beta} = \nabla_{\alpha}\omega_{\beta}$.

6. __Variations of the Stressed State of the Shell with Finite Strains.__ Let $v_{\alpha}{}^*$ be the covariant components of the displacement vector in the coordinate system of the deformed surface, w^* be the projection of this vector on the normal to this surface m^*, and $e_{\alpha\beta}{}^*$ and $\omega_{\alpha}{}^*$ the components of the displacement dyad in the same coordinate system:

$$e_{\alpha\beta}{}^* = \rho_{\beta}{}^* \cdot \frac{\partial v}{\partial x^{\alpha}} = \nabla^*_{\alpha}v_{\beta}{}^* - b_{\alpha\beta}{}^*w^* \tag{6.1}$$

$$\omega_{\alpha}^* = m_* \cdot \frac{\partial v}{\partial x^{\alpha}} = \nabla_{\alpha}w^* + {}^*b_{\alpha}{}^{\gamma}v_{\gamma}{}^*,\ v = \rho_*{}^{\alpha}v_{\alpha}{}^* + m^*w^*$$

Then for components of the first strain tensor, from

$$2p_{\alpha\beta} = \rho_{\alpha}{}^* \cdot \rho_{\beta}{}^* - \rho_{\alpha} \cdot \rho_{\beta} = \rho_{\beta}{}^* \cdot (\rho_{\alpha} + \frac{\partial v}{\partial x^{\alpha}}) -$$

$$\rho_{\alpha} \cdot (\rho_{\beta}{}^* - \frac{\partial v}{\partial x^{\beta}})$$

we have the new expressions

$$2p_{\alpha\beta} = e_{\alpha\beta}{}^* + e_{\beta\alpha} = e_{\beta\alpha}{}^* + e_{\alpha\beta} \tag{6.2}$$

Hence, comparing (6.2) with (1.4), we get

$$e_{\alpha\beta}{}^* = e_{\alpha\beta} + e_\alpha^\lambda e_{\beta\lambda} + \omega_\alpha\omega_\beta \qquad (6.3)$$

and from the equations

$$\nabla_\alpha \omega_\beta{}^* = -\nabla_\alpha (m_* \cdot \rho_\beta) = {}^*b_\alpha^\gamma \rho_\gamma \cdot {}^*\rho_\beta - m_* \cdot \nabla_\alpha \rho_\beta =$$
$$-{}^*b_\alpha^\gamma e_{\beta\gamma}{}^* - m \cdot m_* b_{\alpha\beta} + b_{\alpha\beta}{}^*$$

we get

$$q_{\alpha\beta} = (m \cdot m^* - 1)b_{\alpha\beta} + {}^*b_\alpha{}^\gamma e_{\beta\gamma}{}^* + \nabla_\alpha \omega_\beta{}^* = \qquad (6.4)$$
$$(m \cdot m^* - 1)b_{\alpha\beta} + {}^*b_\alpha^\gamma e_{\beta\gamma}{}^* + \nabla_\alpha{}^*\omega_\beta{}^* + a_*^{\mu\gamma} P_{\gamma,\alpha\beta}\,\omega_\mu{}^*$$

We multiply the vector equilibrium equations (2.5) by the displacement vector v and integrate the result over the entire area of the deformed mean surface

$$\iint_{\sigma^*} v(\nabla_\alpha{}^* T_*{}^\alpha + X_*)\,d\sigma^* = 0$$

or, using the transformation formula

$$\iint_{\sigma^*} \nabla_\alpha{}^*(B \cdot v)\,d\sigma^* = \int_{C^*} B \cdot v\, n_\alpha{}^* ds^* \qquad (6.5)$$

we find

$$\iint_{\sigma^*} X_* \cdot v\, d\sigma^* + \int_{C^*} P \cdot v\, ds^* = \iint_{\sigma^*} (T_*{}^{\alpha\beta} e_{\alpha\beta}{}^* + N_*{}^\alpha \omega_\alpha{}^*)\,d\sigma^*$$

Substituting N_*^α here from (2.4) and again using (6.5) we have

$$\iint_{\sigma^*} (X_* \cdot v + M_*{}^\beta a_*{}^{\lambda\alpha} c_{\beta\lambda}{}^*\omega_\alpha{}^*)\,d\sigma^* + \int_{C^*} (P \cdot v - \qquad (6.6)$$
$$H^{\alpha\beta} n_\alpha{}^*\omega_\beta{}^*)\,ds^* = \iint_{\sigma^*} (T_*{}^{\alpha\beta} e_{\alpha\beta}{}^* - H^{\alpha\beta} \nabla_\alpha{}^*\omega_\beta{}^*)\,d\sigma^*$$

From the expression for the momentum vector $L_* = (m^* \times \rho_\beta{}^*)H^{\alpha\beta} n_\alpha{}^*$ we get

$$H^{\alpha\beta} n_\alpha{}^*\omega_\beta{}^* = G_*\omega_\alpha{}^* n_*{}^\alpha - Hm_* \cdot \frac{dv}{ds^*} \qquad (6.7)$$

Substituting this in (6.6) and taking the function Hw^* to be single valued, we find

$$\iint_{\sigma^*} (X_* \cdot v + M_*^{\beta} a_*^{\lambda\alpha} e_{\beta\lambda}{}^* \omega_\alpha{}^*) d\sigma^* + \int_{C^*} \left\{ \left(P - \frac{\partial Hm^*}{\partial s^*}\right) \cdot v - G_* n_*^{\alpha} \omega_\alpha{}^* \right\} ds^* = \iint_{\sigma^*} (T_*^{\alpha\beta} e_{\alpha\beta}{}^* - H^{\alpha\beta} \nabla_\alpha{}^* \omega_\beta{}^*) d\sigma^* \tag{6.8}$$

or, substituting the symmetric tensors (3.1), (3.10) and (3.11) we get

$$\iint_{\sigma} (X \cdot v + M^{\beta} a_*^{\lambda\alpha} e_{\beta\alpha}{}^* \omega_\alpha{}^*) d\sigma + \int_{C} (\Phi \cdot v - G n^{\alpha} \omega_\alpha{}^*) ds = \iint_{\sigma} \left\{ s^{\alpha\beta} e_{\alpha\beta}{}^* - M^{\alpha\beta} (\nabla_\alpha{}^* \omega_\beta{}^* + {}^*b_\alpha^{\gamma} e_{\beta\gamma}{}^*) \right\} d\sigma \quad \left(G = \sqrt{\frac{a}{a_*}} \frac{ds}{ds^*} G_1\right) \tag{6.9}$$

Here, as before, Φ is the vector of the contour forces taken with respect to a unit length of the undeformed contour C; X and M^{β} are the forces and moments relative to a unit undeformed surface σ. We transform the expression under the integral sign using the right-hand side of (6.8).

Substituting $e_{\alpha\beta}{}^*$ and $\nabla_\alpha{}^* \omega_\beta{}^* + {}^*b_\alpha^{\gamma} e_{\beta\gamma}{}^*$ from (6.3) and (6.4), we get

$$\iint_{\sigma} (X \cdot v + M^{\beta} a_*^{\lambda\alpha} e_{\beta\lambda}{}^* \omega_\lambda{}^*) d\sigma + \int_{C} (\Phi \cdot v - G n^{\alpha} \omega_\alpha{}^*) ds = \iint_{\sigma} \left\{ s^{\alpha\beta} p_{\alpha\beta} - M^{\alpha\beta} q_{\alpha\beta} + M^{\alpha\beta} [(m \cdot m^* - 1) b_{\alpha\beta} + a_*^{\lambda\gamma} p_{\gamma,\alpha\beta} \omega_\lambda{}^*] + \frac{1}{2} s^{\alpha\beta} (e_\alpha^{\lambda} e_{\beta\lambda} + \omega_\alpha \omega_\beta) \right\} d\sigma \tag{6.10}$$

Now we consider the functional

$$\delta A' = \delta \iint \left\{ F + M^{\alpha\beta} [(m \cdot m^* - 1) b_{\alpha\beta} + a_*^{\lambda\gamma} p_{\gamma,\alpha\beta} \omega_\lambda{}^*] + \frac{1}{2} s^{\alpha\beta} (e_\alpha^{\lambda} e_{\beta\lambda} + \omega_\alpha \omega_\beta) \right\} d\sigma \tag{6.11}$$

where F is the deformation potential given by formula (3.16) and

$$\delta A' = \iint_\sigma \{ v \cdot \delta X + \omega_\alpha * \delta(M^\beta a_*^{\alpha\lambda} e_{\beta\lambda}*) + \tag{6.12}$$

$$M^\beta a_*^{\alpha\lambda} e_{\beta\lambda} * \nabla_\alpha v \cdot \delta m_* \} d\sigma + \int_C (v \cdot \delta \Phi -$$

$$\omega_\alpha * n^\alpha \delta G - G n^\alpha \nabla_\alpha v \cdot \delta m_*) ds$$

and we examine under what conditions equation (6.11) holds.

For this purpose we take it in the form

$$\delta A' = \iint_\sigma \{ e_{\alpha\beta} * \delta s^{\alpha\beta} - (\nabla_\alpha * \omega_\beta * + \tag{6.13}$$

$$b^{*\gamma}_\alpha e_{\beta\gamma}*) \delta M^{\alpha\beta} \} d\sigma + \iint_\sigma \{ M^{\alpha\beta} \delta [(m \cdot m^* - 1) b_{\alpha\beta} +$$

$$a_*^{\gamma\lambda} p_{\gamma,\alpha\beta} \omega_\lambda *] + s^{\alpha\beta} (e_{\alpha\lambda} \delta e^\lambda_{\beta\cdot} + \omega_\alpha \delta \omega_\beta) \} d\sigma$$

Here the second integral is equal to:

$$I_2 = \iint_\sigma \{ M^{\alpha\beta} [\omega_\gamma * \delta(a_*^{\gamma\lambda} p_{\lambda,\alpha\beta}) - \nabla_\alpha * \nabla_\beta v \cdot \delta m_*] + \tag{6.14}$$

$$s^{\alpha\beta} \nabla_\alpha v \cdot \delta \rho_\beta * \} d\sigma$$

since

$$e_{\alpha\lambda} \delta e^\lambda_{\beta\cdot} + \omega_\alpha \delta \omega_\beta = \nabla_\alpha v \cdot \delta \rho_\beta *$$

$$m b_{\alpha\beta} \cdot \delta m_* + a_*^{\gamma\lambda} p_{\lambda,\alpha\beta} \delta \omega_\gamma * = - \nabla_\alpha * \nabla_\beta v \cdot \delta m_*$$

Now on the basis of the equilibrium equations (3.12), (3.13), and (3.14), on integrating by parts using (6.5) we get

$$I_2 = \iint_\sigma (X^\beta v \cdot \delta \rho_\beta * + X^3 v \cdot \delta m_* + \nabla_\beta v \cdot \delta m_* \cdot \tag{6.15}$$

$$a_*^{\gamma\beta} e_{\alpha\gamma} * M^\alpha) d\sigma + \int_C (v \Phi^\alpha \cdot \delta \rho_\alpha * + v \Phi^3 \cdot \delta m_* -$$

$$n^\alpha \nabla_\alpha v \cdot G \delta m_* + v_\gamma * H_2 \tau^\alpha \delta b^{*\gamma}_\alpha) ds + \iint_\sigma \{ v_\beta * [Q^\alpha \delta * b^\beta_\alpha -$$

$$(s^{\alpha\gamma} - * b^\gamma_\lambda M^{\lambda\alpha}) \delta(a_*^{\beta\nu} p_{\nu,\alpha\gamma})] - w^*(s^{\alpha\beta} -$$

$$* b^\beta_\gamma M^{\gamma\alpha}) \delta b_{\alpha\beta} * + M^{\alpha\beta} \omega_\gamma * \delta(a_*^{\gamma\nu} p_{\nu,\alpha\beta}) \} d\sigma$$

The first integral in (6.13) agrees in form with the right-hand side of (6.8). Therefore on integrating by parts from I_1 we get the equilibrium equations and the boundary conditions with the variation of forces and moments, when the coefficients of the first and second quadratic forms of the deformed surface are constant. Assuming this and substituting (6.15) in (6.13), we finally get

$$\int_C \Big\{ v_\alpha{}^* \delta(\Phi^\alpha - T_1{}^{\alpha\beta} n_\beta - {}^*b^\alpha_\beta \tau^\beta H_2) + w^* \delta(\Phi_3 - Q^\alpha n_\alpha + \frac{\partial H_2}{\partial s}) - \omega_\alpha {}^*n^\alpha \delta(G - \sqrt{\frac{a_*}{a}} \frac{ds}{ds^*} M^{\alpha\beta} n_\alpha n_\beta) \Big\} ds = \tag{6.16}$$

$$\iint_\sigma \Big\{ \omega_\beta{}^* [\nabla_\alpha \delta M^{\alpha\beta} + \delta(a_*{}^{\beta\gamma} P_{\gamma,\alpha\lambda} M^{\alpha\lambda}) - \delta Q^\beta - \delta(M^\alpha{}_{a_*}{}^{\beta\lambda} c_{\alpha\lambda}{}^*)] - v_\beta{}^* [\nabla_\alpha \delta T_1{}^{\alpha\beta} + \delta(a_*{}^{\beta\lambda} P_{\gamma,\alpha\lambda} T_1{}^{\alpha\lambda}) - \delta({}^*b^\beta_\alpha Q^\alpha) + \delta X^\beta] - w^* [\nabla_\alpha \delta Q^\alpha + \delta(b_{\alpha\beta} {}^*T_1{}^{\alpha\beta}) + \delta X^3] \Big\} d\sigma$$

where

$$T_1{}^{\alpha\beta} = s^{\alpha\beta} - {}^*b^\beta_\gamma M^{\gamma\alpha}$$

Thus, the variational equation (6.11) is valid if:

(a) the displacement dyad is varied*

* Note: The variations of the components of the dyad are connected by the relations[11]

$$c^{\beta\gamma} \nabla_\gamma \delta e_{\beta\alpha} = c^{\beta\gamma} b_{\gamma\alpha} \delta\omega_\beta, \quad c^{\alpha\beta} \nabla_\alpha \delta\omega_\beta = c^{\alpha\beta} b^\gamma_\beta \delta e_{\alpha\gamma}$$

(b) the variations of the forces and moments satisfy the equilibrium equations

$$\nabla_\alpha \delta T_1{}^{\alpha\beta} + \delta(a_*{}^{\beta\gamma} P_{\gamma,\alpha\lambda} T_1{}^{\alpha\lambda}) - \delta({}^*b^\beta_\alpha Q^\alpha) + \delta X^\beta = 0 \tag{6.17}$$

$$\nabla_\alpha \delta Q^\alpha + \delta(b_{\alpha\beta} {}^*T_1{}^{\alpha\beta}) + \delta X^3 = 0 \tag{6.18}$$

$$\nabla_\alpha \delta M^{\alpha\beta} + \delta(a_*{}^{\beta\gamma} P_{\gamma,\alpha\lambda} M^{\alpha\lambda}) - \delta Q^\beta - \delta(M^\alpha{}_{a_*}{}^{\beta\lambda} c_{\alpha\lambda}) = 0 \tag{6.19}$$

(c) on the contour of the undeformed surface the variations of the forces and moments satisfy the static boundary conditions

$$\delta \Phi^{\alpha} = n_{\beta} \delta T_1^{\alpha\beta} + \tau^{\beta} \delta ({}^{*}b_{\beta}^{\alpha} H_2), \quad \delta \Phi_3 = n_{\alpha} \delta Q^{\alpha} - \frac{\partial \delta H_2}{\partial s}, \quad \delta G = n_{\alpha} n_{\beta} \delta \left(\sqrt{\frac{a*}{a}} \frac{ds}{ds*} M^{\alpha\beta}\right) \tag{6.20}$$

In equations (6.17), (6.18) and (6.19) the coefficients $a_{*}^{\lambda\gamma}$ $P_{\gamma,\alpha\beta}$ and ${}^{*}b_{\alpha}^{\beta}$ are varied since they depend on the forces and the moments.

In the case of a mean deflection, equation (6.11) is simplified:

$$\iint_{\sigma} (v \cdot \delta X + \omega_{\alpha} a^{\alpha\lambda} e_{\beta\lambda} \delta M^{\beta}) d\sigma + \int_{C} (v \cdot \delta \Phi - \omega_{\alpha} n^{\alpha} \delta G) ds = \frac{1}{2} \delta \iint_{\sigma} (s^{\alpha\beta} p_{\alpha\beta} - M^{\alpha\beta} q_{\alpha\beta} + \omega_{\alpha} \omega_{\beta} s^{\alpha\beta}) d\sigma \tag{6.21}$$

since in this case

$$\omega_{\alpha}^{*} = \omega_{\alpha}, \quad 2F = s^{\alpha\beta} p_{\alpha\beta} - M^{\alpha\beta} q_{\alpha\beta}, \quad mm_{*} - 1 \sim e_{\alpha\lambda} e_{\beta}^{\lambda} \sim \varepsilon_p^{2}$$

We assume that $\delta X = 0$, $M = 0$ and that on the contour of the undeformed shell the following condition is satisfied

$$\int_{C} \left(v \cdot \delta \Phi - \omega_{\alpha} {}^{*}n^{\alpha} \delta G - G n^{\alpha} \frac{\partial v}{\partial x^{\alpha}} \cdot \delta m_{*}\right) ds = 0 \tag{6.22}$$

Then this theorem holds: the actual stressed state of a shell differs from all statically possible states in that for it the functional

$$R = \iint_{\sigma} \left\{ F + M^{\alpha\beta} \left[(m \cdot m^{*} - 1) b_{\alpha\beta} + a_{*}^{\lambda\gamma} G_{\gamma,\alpha\beta} \omega^{*}_{\lambda} \right] + \frac{1}{2} s^{\alpha\beta} (e_{\alpha\lambda} e_{\beta}^{\lambda} + \omega_{\alpha} \omega_{\beta}) \right\} d\sigma \tag{6.23}$$

has a stationary value; i.e., $\delta R = 0$.

This theorem also expresses the origin of possible changes of the stressed state in the nonlinear theory of shells.

In the case of infinitely small displacements the stationary value is a minimum, and $\delta R = 0$ expresses Castigliano's principle.

The contour condition (6.22) is satisfied if, for example:

(1) the contour is free

$$\Phi^{\alpha} = \Phi^{3} = G = 0 \tag{6.24}$$

(2) the contour is rigidly fixed

$$v = \omega_{\alpha} n^{\alpha} = 0 \quad (\delta m^{*} = \delta m = 0) \tag{6.25}$$

(3) part of the contour is free, and another part is fixed;

(4) the contour has a motionless hinged support

$$v = 0, \quad G = 0 \tag{6.26}$$

(5) the contour is freely supported

$$v = 0, \quad \Phi^{\alpha} = G = 0 \tag{6.27}$$

The equation $\delta R = 0$ is the variational formulation of the conditions of continuity for finite deformations of the shell.

In fact, putting

$$S^{\alpha\beta}\rho_{\beta} + M^{\alpha\beta} m_{\beta}^{*} + m^{*}Q^{\alpha} = c^{\alpha\beta}\Delta_{\beta}\varphi$$

where φ is the vector stress function, the equilibrium equations (3.12) and (3.13) with $X = M = 0$ also satisfy, as a result of the contour condition (6.22), the equation

$$\delta R = -\iint_{\sigma} c^{\alpha\beta}\nabla_{\beta}\rho_{\alpha} \cdot{}^{*} \delta\varphi = 0$$

Expanding this with the help of (1.10), the Bianchi formulas, and the Ricci identities, we find the conditions for the continuity of finite deformations:

$$c^{\beta\gamma}c^{\sigma\alpha}\left(\nabla_{\gamma}\nabla_{\alpha}p_{\sigma\beta} - \frac{1}{2}q_{\alpha\beta}q_{\gamma\sigma} + \frac{1}{2}a_{*}^{\pi\lambda}p_{\pi,\alpha\beta}p_{\lambda,\sigma\gamma}\right) -$$

$$(2Ha^{\alpha\beta} - b^{\alpha\beta})q_{\alpha\beta} - Ka^{\alpha\beta}p_{\alpha\beta} = 0$$

$$c^{\beta\gamma}\left\{\nabla_\gamma q_{\alpha\beta} - a_*^{\sigma\lambda}(b_{\lambda\gamma} + q_{\lambda\gamma})F_{\sigma,\alpha\beta} = 0\right.$$

where H and K are the mean and Gaussian curvature of the undeformed surface.

We assume that there is no question of a loss of accuracy in the problem considered. Then it is possible to neglect the products $S^{\alpha\beta} e_{\alpha\lambda} e_{\beta}^{\lambda}$, and the vector function ψ and the two rotation angles ω_α are permitted to vary in the functional R. If, moreover, the shell is sloped or if w is a rapidly varying function (local loss of stability, edge effect), the terms $b_\alpha^\lambda v_\lambda$ are small with respect to $\nabla_\alpha w$ and as a result $w_\alpha \approx \nabla_\alpha w$.

In this case the functions ψ and w are permitted to vary (cf. the work of N. A. Alumyae[1]). In the general case, as distinguished from the linear theory, the variation of the stressed state of the shell is accompanied by variations of the three rotation angles: $\Omega = \frac{1}{2} c^{\alpha\beta} e_{\alpha\beta}$ and ω_α. This makes it difficult to apply Castigliano's principle to real problems. But in the case of homogeneous equations the parameters $e_{\alpha\beta}$ and ω_α can be left out.

7. <u>The Variational Formula Corresponding to Homogeneous Equilibrium Equations</u>. We assume that X = M = 0 and consider the integral

$$I_\sigma = \iint_\sigma \left\{ \frac{1}{2} \nabla_\alpha V \cdot \nabla_\beta V \cdot S^{\alpha\beta} + M^{\alpha\beta} \left[(m \cdot m^* - 1) b_{\alpha\beta} + \right.\right. \qquad (7.1)$$

$$\left.\left. a_*^{\lambda\gamma} P_{\gamma,\alpha\beta}\, \omega_\lambda^* \right] \right\} d\sigma$$

Substituting in this in place of $S^{\alpha\beta}$ and $M^{\alpha\beta}$ their expression in terms of stress functions (3.24) and (3.25) and integrating the result by parts, we get

$$I_\sigma = I_C - \iint_{\sigma^*} \left\{ \psi_\gamma c_*^{\alpha\pi} c_*^{\beta\gamma} \left[\nabla_\pi^* (m \cdot m^* b_{\alpha\beta} - \right.\right. \qquad (7.2)$$

$$b_{\alpha\beta} + a_*^{\lambda\nu} P_{\nu,\alpha\beta}\, \omega_\lambda^*) + {}^*b_\alpha^\lambda \nabla_\lambda V \cdot \nabla_\pi^* \nabla_\beta V \Big] +$$

$$\psi\, c_*^{\alpha\pi} c_*^{\beta\gamma} \left[(m \cdot m^* - 1) b_{\alpha\beta} + a_*^{\lambda\nu} P_{\nu,\alpha\beta}\, \omega_\lambda^* + \right.$$

$$\left.\left. \frac{1}{2} {}^*b_\alpha^\lambda \nabla_\beta V \cdot \nabla_\lambda V \right] b_{\pi\gamma}^* \right\} d\sigma^*$$

$$I_C = \int_{C^*} c_*^{\alpha\pi} c_*^{\beta\gamma} \Big\{ \tfrac{1}{2} \nabla_\alpha V \cdot \nabla_\beta V \cdot (\nabla_\gamma \psi + {}^*b_\gamma^\sigma \psi_\sigma) + \qquad (7.3)$$

$$\tfrac{1}{2} \psi_\gamma {}^*b_\alpha^\lambda \nabla_\lambda V \cdot \nabla_\beta V + \psi_\gamma \left[(m \cdot m^* - 1) b_{\alpha\beta} + a_*^{\lambda\nu} P_{\nu,\alpha\beta} \omega_\lambda^*\right]$$

$$D_\pi^* ds^*$$

In order to transform the double integral in (7.2) we note that

$$(m \cdot m^* - 1) b_{\alpha\beta} + a_*^{\lambda\nu} P_{\nu,\alpha\beta} \omega_\lambda^* + \tfrac{1}{2} {}^*b_\alpha^\lambda \; \nabla_\beta V \cdot \nabla_\lambda V = \qquad (7.4)$$

$$q_{\alpha\beta} - \nabla_\alpha^* \omega_\beta^* - P_{\alpha\beta}$$

$$A_{\pi\alpha\beta} = \nabla_\pi^* \left[(m \cdot m^* - 1) b_{\alpha\beta} + a_*^{\lambda\nu} P_{\nu,\alpha\beta} \omega_\lambda^*\right] +$$

$$ {}^*b_\alpha^\lambda \nabla_\lambda V \cdot \nabla_\pi^* \nabla_\beta V = \nabla_\pi^* \left[(m^* - m) \cdot \nabla_\alpha^* \rho_\beta\right] +$$

$$ {}^*b_\alpha^\lambda \nabla_\lambda V \cdot \nabla_\pi^* (\rho_\beta^* - \rho_\beta) = (m^* - m) \cdot$$

$$\nabla_\pi^* \nabla_\alpha^* \rho_\beta + \nabla_\alpha^* \rho_\beta \cdot (b_\pi^\lambda \rho_\lambda - {}^*b_\pi^\lambda \rho_\lambda^*) +$$

$$ {}^*b_\alpha^\lambda \nabla_\lambda V \cdot (b_{\pi\beta}{}^* m_* - b_{\pi\beta} m + a_*^{\mu\nu} P_{\nu,\pi\beta} \rho_\mu) =$$

$$(m^* - m) \cdot \nabla_\pi^* \nabla_\alpha^* \rho_\beta + (b_{\alpha\beta} m - \rho_\nu a_*^{\nu\mu} P_{\mu,\alpha\beta}) \cdot$$

$$(b_\pi^\lambda \rho_\lambda - {}^*b_\pi^\lambda \rho_\lambda^*) + {}^*b_\alpha^\lambda (b_{\pi\beta}^* \omega_\lambda^* - b_{\pi\beta} \omega_\lambda +$$

$$a_*^{\mu\nu} P_{\nu,\pi\beta}^* e_{\lambda\mu}) = (m^* - m) \cdot \nabla_\pi^* \nabla_\alpha^* \rho_\beta -$$

$$\omega_\lambda (b_{\alpha\beta} {}^*b_\pi^\lambda + b_{\pi\beta} {}^*b_\alpha^\lambda) + a_*^{\mu\nu} P_{\mu,\alpha\beta} (q_{\pi\nu} -$$

$$ {}^*b_\pi^\lambda e_{\nu\lambda}^*) + a_*^{\mu\nu} P_{\nu,\pi\beta} {}^*b_\alpha^\lambda e_{\lambda\mu} + {}^*b_\alpha^\lambda b_{\pi\beta}^* \omega_\lambda^*$$

Noting that

$$c_*^{\beta\gamma} c_*^{\alpha\pi} \nabla_\pi^* \nabla_\alpha^* \rho_\beta \cdot (m^* - m) = \tfrac{1}{2} c_*^{\beta\gamma} c_*^{\alpha\pi} R^*{}_{\pi\alpha\beta}^{\;\lambda} \, \rho_\lambda \cdot$$

$$(m^* - m) = -c_*^{\alpha\pi} c_*^{\beta\gamma} b_{\pi\beta}^* {}^*b_\alpha^\lambda \omega_\lambda^*$$

$$c_*^{\alpha\pi} c_*^{\beta\gamma} \omega_\lambda (b_{\alpha\beta} {}^*b_\pi^\lambda + b_{\pi\beta} {}^*b_\alpha^\lambda) = 0$$

$$c_*^{\alpha\pi} c_*^{\beta\gamma} (-a_*^{\mu\nu} P_{\mu,\alpha\beta} {}^*b_\pi^\lambda e_{\gamma\lambda}^* + a_*^{\mu\nu} P_{\nu,\pi\beta} {}^*b_\alpha^\lambda e_{\lambda\mu}) =$$

$$2 c_*^{\alpha\pi} c_*^{\beta\gamma} {}^*b_\beta^\sigma a_*^{\lambda\mu} P_{\mu,\alpha\gamma} D_{\lambda\sigma}$$

and that

$$c_*^{\alpha\pi} c^{\beta\pi} A_{\pi\alpha\beta}\psi_\gamma = c_*^{\alpha\pi} c_*^{\beta\gamma} (q_{\pi\nu} p_{\mu,\alpha\beta} + \qquad (7.5)$$
$$2p_{\gamma\sigma}{}^* b_\beta^\sigma p_{\mu,\alpha\gamma}) a_*^{\mu\nu}\psi_\gamma$$

Then the integral (7.2) on taking (7.4) and (7.5) into account assumes the form:

$$I_\sigma = I_{C*} - \iint_{\sigma^*} c_*^{\alpha\pi} c_*^{\beta\gamma} \left[(q_{\pi\nu} - 2p_{\nu\sigma}{}^* b_\pi^\sigma) a_*^{\mu\nu} p_{\mu,\alpha\beta}\psi_\gamma + \qquad (7.6)\right.$$
$$\left.\psi b_{\pi\gamma}{}^*(q_{\alpha\beta} - \nabla_\alpha{}^*\omega_\beta{}^* - p_{\alpha\beta})\right] d\sigma^*$$

In order to eliminate $\nabla_\alpha{}^*\omega_\beta{}^*$ from this, we consider the integral

$$I = \iint_{\sigma^*} c_*^{\alpha\pi} c_*^{\beta\gamma}\psi\nabla_\gamma{}^*\nabla_\pi{}^*p_{\alpha\beta}\, d\sigma^* =$$
$$\iint_{\sigma^*} c_*^{\alpha\pi} c_*^{\beta\gamma}\psi\nabla_\gamma{}^*\nabla_\pi{}^*(e_{\alpha\beta}{}^* - \frac{1}{2}\nabla_\alpha V\cdot\nabla_\beta V)\, d\sigma^* =$$
$$\iint_{\sigma^*} c_*^{\alpha\pi} c_*^{\beta\gamma}\psi\nabla_\gamma{}^*\nabla_\pi{}^*e_{\alpha\beta}{}^*\, d\sigma^*$$

On account of the identities $c_*^{\alpha\pi}\nabla_\pi{}^*e_{\alpha\beta}{}^* = c_*^{\alpha\pi} b_{\pi\beta}{}^*\omega_\alpha{}^*$ it is equal to:

$$I = \iint_{\sigma^*} c_*^{\alpha\pi} c_*^{\beta\gamma}\psi b_{\pi\beta}{}^*\nabla_\gamma{}^*\omega_\alpha{}^*\, d\sigma^*$$

Substituting this in (7.6), we have

$$I_\sigma = I_{C*} - \iint_{\sigma^*} c_*^{\alpha\pi} c_*^{\beta\gamma} \left[(q_{\pi\lambda} - 2p_{\lambda\sigma}{}^* b_\pi^\sigma) a_*^{\mu\lambda} p_{\mu,\alpha\beta}\psi_\gamma + \right.$$
$$\left.\psi(b_{\pi\gamma}{}^*q_{\alpha\beta} - b_{\pi\gamma}{}^*p_{\alpha\beta} + \nabla_\gamma{}^*\nabla_\pi{}^*p_{\alpha\beta})\right] d\sigma^*$$

or, introducing here instead of $c_*^{\alpha\pi} c_*^{\beta\gamma}\nabla_\gamma{}^*\nabla_\pi{}^*p_{\alpha\beta}$ its value taken from the condition of continuity of deformation referred to the coordinate system of the deformed surface [11]:

$$c_*^{\alpha\pi} c_*^{\beta\gamma} (\nabla_\gamma{}^*\nabla_\pi{}^*p_{\alpha\beta} - \frac{1}{2}q_{\pi\beta}q_{\alpha\gamma} - \frac{1}{2}b_{\alpha\gamma}{}^*q_{\pi\beta} - \qquad (7.7)$$
$$\frac{1}{2}q_{\alpha\gamma}b_{\pi\beta}{}^* + \frac{1}{2}a^{\lambda\nu}p_{\lambda,\pi\beta}{}^*p_{\nu,\alpha\gamma}{}^*) = K^*a_*^{\alpha\beta}p_{\alpha\beta}$$

where $a^{\lambda\nu} P_{\nu,\alpha\gamma}{}^* = a_*^{\lambda\nu} P_{\nu,\alpha\gamma}$, $K = \det({}^*b^\sigma_\beta)$, we find

$$I_\sigma = I_{\sigma*} - \iint_{\sigma*} c^{\alpha\pi} c^{\beta\gamma} [(q_{\pi\lambda} - 2p_{\lambda\sigma}\, {}^*b^\sigma_\pi) a_*{}^{\mu\lambda} \tag{7.8}$$

$$P_{\mu,\alpha\beta} \psi_\gamma' + \frac{1}{2} \psi' (q_{\pi\gamma} q_{\alpha\beta} - a_*{}^{\lambda\nu} a_*{}^{\mu\sigma} a_{\mu\nu} P_{\lambda,\pi\beta}$$

$$P_{\sigma,\alpha\gamma})] d\sigma \qquad (\psi' = \psi \sqrt{\frac{a}{a^*}}, \quad \psi_\gamma' = \psi_\gamma \sqrt{\frac{a}{a^*}})$$

We now transform the contour integral (7.3). Recalling the formula (6.4) and the formula $2(e_{\alpha\beta}{}^* - P_{\alpha\beta}) = (\nabla_\alpha V)(\nabla_\beta V)$, we have

$$I_C = -\int_{C^*} c_*{}^{\alpha\pi} c_*{}^{\beta\gamma} [(\nabla_\gamma \psi + {}^*b^\sigma_\gamma \psi_\sigma) P_{\alpha\beta} + \psi_\gamma\, {}^*b^\lambda_\alpha P_{\beta\lambda} - \tag{7.9}$$

$$q_{\alpha\beta} \psi_\gamma]\, n_\pi{}^* ds^* + \int_{C^*} c_*{}^{\alpha\pi} c_*{}^{\beta\gamma} [e_{\alpha\beta}{}^* (\nabla_\gamma \psi +$$

$${}^*b^\sigma_\gamma \psi_\sigma) - \psi_\sigma \nabla_\alpha{}^* \omega_\beta{}^*] n_\pi{}^* ds^*$$

To satisfy the boundary conditions (6.24)-(6.27) from (6.9) we get

$$\iint_{\sigma*} c_*{}^{\alpha\pi} c_*{}^{\beta\gamma} [\nabla_\pi{}^* (\nabla_\gamma \psi + {}^*b^\sigma_\gamma \psi_\sigma) e_{\alpha\beta}{}^* -$$

$$\nabla_\alpha{}^* \omega_\beta{}^* (\nabla_\pi{}^* \psi_\gamma - b_{\pi\gamma}{}^* \psi)] d\sigma^* = 0$$

From this, integrating by parts we get

$$\int_{C^*} c_*{}^{\alpha\pi} c_*{}^{\beta\gamma} [(\nabla_\gamma \psi + {}^*b^\sigma_\gamma \psi_\sigma) e_{\alpha\beta}{}^* n_\pi{}^* - \omega_\beta{}^* (\nabla_\pi{}^* \psi_\gamma -$$

$$b_{\pi\gamma}{}^* \psi) n_\alpha{}^*] ds^* = 0$$

This makes it possible to represent the second integral in (7.9) in the form

$$I_{C'} = \int_{C^*} c_*{}^{\alpha\pi} c_*{}^{\beta\gamma} [e_{\alpha\beta}{}^* (\nabla_\gamma \psi + {}^*b^\sigma_\gamma \psi_\sigma) - \psi_\gamma \nabla_\alpha{}^* \omega_\beta{}^*] n_\pi{}^* ds^* =$$

$$\text{const} - \int_{C^*} c_*{}^{\alpha\pi} c_*{}^{\beta\gamma} \omega_\beta\, {}^*b_{\pi\gamma} \psi\, n_\alpha{}^* ds^*$$

Furthermore, since $c_*^{\alpha\pi} c_*^{\beta\gamma} b_{\pi\gamma}\,{}^*\omega_\beta{}^* =$
$c_*^{\alpha\pi} c_*^{\beta\gamma} \nabla_\gamma\, {}^*p_{\beta\pi}{}^* = c_*^{\alpha\pi} c_*^{\beta\gamma} \nabla_\gamma\, {}^*p_{\beta\pi}$

$$I_{C'} = \text{const} - \int_{C^*} c_*^{\alpha\pi} c_*^{\beta\gamma} \psi \nabla_\gamma\, {}^*p_{\beta\pi}\, n_\alpha\, {}^*ds^*$$

Then instead of (7.9) we get

$$I_C = -\int_C c^{\alpha\pi} c^{\beta\gamma} [(\nabla_\gamma \psi' + {}^*b_\gamma^\sigma\, \psi_\sigma{}')p_{\alpha\beta} - \psi' \nabla_\gamma\, {}^*p_{\alpha\beta} + \quad (7.10)$$
$$\psi_\gamma{}'\, {}^*b_\alpha^\lambda\, p_{\lambda\beta} - q_{\alpha\beta}\psi_\gamma{}']n_\pi\, ds$$

Thus, the functional R is transformed to the form

$$R = -\int_C c^{\alpha\pi} c^{\beta\gamma} [(\nabla_\gamma \psi' + {}^*b_\gamma^\sigma\, \psi_\sigma{}')p_{\alpha\beta} - \psi' \nabla_\gamma\, {}^*p_{\alpha\beta} +$$
$$\psi_\gamma{}'\, {}^*b_\alpha^\lambda\, p_{\lambda\beta} - q_{\alpha\beta}\psi_\gamma{}']n_\pi\, ds - \iint_\sigma c^{\alpha\pi} c^{\beta\gamma} [(q_{\pi\lambda} -$$
$$2p_{\lambda\sigma}\, {}^*b_\pi^\sigma) a_*^{\mu\lambda} p_{\mu,\alpha\beta}\, \psi_\gamma{}' + \frac{1}{2}\psi'(q_{\pi\gamma} q_{\alpha\beta} -$$
$$a_*^{\lambda\nu} a_*^{\mu\sigma} a_{\mu\nu} p_{\nu\pi\beta}\, p_{\sigma,\alpha\gamma})]d\sigma + \iint_\sigma F\, d\sigma$$

Here the quantities $p_{\alpha\beta}$ and $q_{\alpha\beta}$ can be expressed by means of ψ and ψ_α to the desired accuracy.

Consequently, in the equilibrium state the functional (7.11) has a stationary value when the static boundary conditions are satisfied with respect to ψ and ψ_α and also the supplementary condition (6.22). From the stationary condition $\delta R = 0$ there results a system of three differential equations for the functions ψ and ψ_α (conditions of continuity of deformations expressed in terms of ψ and ψ_α) and also natural boundary conditions for the stress functions.

For small deformations (7.11) is simplified and results in the form

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